

# SINGULAR DISTRIBUTIONS AND SYMMETRY OF THE SPECTRUM

GADY KOZMA AND ALEXANDER OLEVSKII

This is a survey of the “Fourier symmetry” of measures and distributions on the circle in relation with the size of their support. Mostly it is based on our paper [16] and a talk given by the second author in the 2012 Abel symposium.

## 1. INTRODUCTION

Below we denote by  $S$  a Schwartz distribution on the circle group  $\mathbb{T}$ ; by  $K_S$  its support; and by  $\widehat{S}(n)$  its Fourier transform. It has polynomial growth.  $S$  is called a *pseudo-function* if  $\widehat{S}(n) = o(1)$ . In this case the Fourier series

$$\sum_{n=-\infty}^{\infty} \widehat{S}(n) e^{int}$$

converges to zero pointwisely on  ${}^c K$ , see [9].

The purpose of this survey is to highlight the phenomenon that some fundamental properties of a trigonometric series depend crucially on the “balance” between its analytic and anti-analytic parts (which correspond to the positive and negative parts of the spectrum). We start with a classic examples, comparing Menshov’s and Privalov’s theorems.

**Theorem** (Menshov, [20]). *There is a (non-trivial) singular, compactly supported measure  $\mu$  on the circle which is a pseudo-function.*

See [1, §XIV.12]. In particular, it means that a non-trivial trigonometric series

$$\sum_{n \in \mathbb{Z}} c(n) e^{int} \tag{1}$$

may converge to zero almost everywhere. This disproved a common belief that the uniqueness results of Riemann and Cantor may be strengthened to any trigonométrique series converging to zero almost everywhere (shared by Lebesgue, [17]). Contrast Menshov’s theorem with the following result:

**Theorem** (Abel & Privalov). *An “analytic” series*

$$\sum_{n \geq 0} c(n) e^{int} \tag{2}$$

*can not converge to zero on a set of positive measure, unless it is trivial.*

The theorem of Abel [26, §3.14] gives that if  $\sum c(n) e^{int}$  converges at some  $t$ , then the function  $\sum c(n) z^n$ , which is analytic in the disc  $\{|z| < 1\}$ , converges *non-tangentially* to the same value at  $e^{it}$ . The theorem of Privalov [12, §D.III] claims that an analytic function on the disc which converges non-tangentially to zero on a set of positive measure is identically zero. Together these two results give the theorem above.

We are interested in different aspects of symmetry and non-symmetry of the Fourier transform of measures and distributions. Our first example is Frostman's theorem.

## 2. FROSTMAN'S THEOREM

The classic Frostman theorem connects the Hausdorff dimension of a compact set  $K$  to the (symmetric) behaviour of the Fourier coefficients of measures supported on  $K$ . Let us state it in the form relevant to us.

**Theorem** (Frostman).

(i) *If a compact  $K$  supports a probability measure  $\mu$  s.t.*

$$\sum_{n \neq 0} \frac{|\widehat{\mu}(n)|^2}{|n|^{1-a}} < \infty \quad (3)$$

*then  $\dim K \geq a$ .*

(ii) *If  $\dim K > a$  then  $K$  supports a probability measure  $\mu$  satisfying (3).*

In fact, the first clause may be strengthened to only require that  $K$  supports a (non-trivial) complex-valued measure, or even a distribution satisfying (3). This follows from the following result

**Theorem** (Beurling). *If  $K$  supports a distribution  $S$  satisfying (3) then it also supports a probability measure with this property.*

see [9, Théorème V, §III] or [3]. It is worth contrasting this with the result of Piatetski-Shapiro [22] that one may find a compact  $K$  which supports a distribution  $S$  with  $\widehat{S}(n) \rightarrow 0$  but does not support a measure  $\mu$  with  $\widehat{\mu}(n) \rightarrow 0$ . We will return to the theorem of Piatetski-Shapiro in section 6.

The first result from [16] which we wish to state is the one-sided version of the theorem of Frostman & Beurling:

**Theorem 1.** *If  $K$  supports a distribution  $S$  such that*

$$\sum_{n < 0} \frac{|\widehat{S}(n)|^2}{|n|^{1-a}} < \infty$$

*then  $\dim(K) \geq a$ .*

*Proof sketch, step 1.* Examine first the case that  $a = 1$ . Then  $S$  is a distribution with anti-analytic part belonging to  $L^2(\mathbb{T})$ . Based on some Phragmén-Lindlöf type theorems in the disc due to Dahlberg [4] and Berman [2], one can prove that if  $\dim K < 1$  then  $S = 0$ .

*Step 2.* We now reduce the case of general  $a$  to the case of  $a = 1$ . We use Salem's result which gives a quantitative estimate of the Menshov theorem above. Namely, given  $d$ ,  $0 < d < 1$ , there is a probability measure  $\nu$  supported by a compact set  $E$  of Hausdorff dimension  $< d + \epsilon$ , and such that  $\widehat{\nu}(n) \leq C|n|^{-d/2}$ . Notice that, as in the original result of Menshov, not every "thick" compact supports such a measure. Some arithmetics of  $E$  is involved (this is in stark contrast to Frostman's theorem where only the dimension plays a role). Take  $d = 1 - a$  and convolve:  $S' = S * \nu$ . Then  $S'$  has its antianalytic part in  $L^2$ . Thus  $\dim(K + E) = 1$ . The theorem is proved using the inequality  $\dim(S + E) \leq \dim S + \dim E$ .

A difficulty is that it is not true in general that  $\dim(A + B) \leq \dim A + \dim B$ . A counterexample may be constructed by making  $A$  large in some “scales” and small in others, and making  $B$  large in the scales where  $A$  is small and vice versa. Since the Hausdorff dimension is determined by the best scales, both  $A$  and  $B$  would have small Hausdorff dimension but  $A + B$  would be large in all scales, and hence would have large Hausdorff dimension. It is even possible to achieve  $\dim A = \dim B = 0$  while  $\dim(A + B) = 1$  [5, example 7.8]. However, if  $\dim A$  is understood in the stronger sense of upper Minkowski, or box dimension (which requires smallness in *all* scales), then the inequality holds. As all the standard constructions of Salem sets in fact have the same Hausdorff and Minkowski dimensions, the theorem is finished.  $\square$

Let us remark that this proof is quite different from the original proofs of Frostman and Beurling, which were potential-theoretic in nature.

### 3. “ALMOST ANALYTIC” SINGULAR PSEUDO-FUNCTIONS.

There is a delicate difference between symmetric and one-sided situations. The Frostman-Beurling condition in fact implies that compact  $K$  has positive  $a$ -measure, while the one-sided version does not, at least for  $a = 1$ . The latter was proved in [14]

**Theorem 2.** *There is a distribution  $S$  with the properties:*

- (i)  $\widehat{S}(n) = o(1)$ .
- (ii)  $m(\text{supp } S) = 0$ .
- (iii)  $\sum_{n < 0} |\widehat{S}(n)|^2 < \infty$ .

One can say: a singular pseudo-function can be “almost” analytic, in the sense that its antianalytic part is a usual  $L^2$  function (notice that a singular distribution is never analytic, i.e. the antianalytic part cannot be empty). It might be interesting to compare this with classical Riemannian uniqueness theory, which teaches us that

uniqueness of the decomposition of  $f$  in trigonometric series implies Fourier formulas for coefficients.

To make this precise, recall that a set  $K$  for which no non-trivial series may converge to zero outside  $K$  is called a set of uniqueness, or a  $\mathcal{U}$ -set. With this notation we have

**Theorem (du Bois-Reymond, Privalov).** *If a finite function  $f \in L^1(\mathbb{T})$  has a decomposition in a series (1), which converges on  $\mathbb{T} \setminus K$  for some compact  $\mathcal{U}$ -set  $K$  then the series is the Fourier expansion of  $f$ .*

See [26, theorem IX.6.19] or [23], and [1, Chap. I, §72 and Chap. XIV, §4] for the history of the theorem, and a version for  $K$  countable but not necessarily compact.

In contrast, take  $S$  from theorem 2. Then

$$\sum \widehat{S}(n)e^{int} = 0 \quad \forall t \in \mathbb{T} \setminus K.$$

Consider the “anti-analytic” part  $f := \sum_{n < 0} \widehat{S}(n)e^{int}$ . Then  $f$  is an  $L^2$  function on  $\mathbb{T}$ , smooth on  $\mathbb{T} \setminus K$ . It admits an “analytic decomposition”:

$$f(t) = \sum_{n \geq 0} c(n) e^{int} \quad c(n) := -\widehat{S}(n)$$

which converges pointwisely there. Such a representation is *unique*, however it is *not* the Fourier one.

#### 4. CRITICAL SIZE OF THE SUPPORT OF “ALMOST ANALYTIC” DISTRIBUTIONS

One can say a bit more about the support of “one-sided Frostamn’s distributions”. Let  $h(t) := t \log 1/t$ , and  $\Lambda_h$  the corresponding Hausdorff measure.

**Theorem 3.** *If  $S$  is a (non-trivial) distribution such that the anti-analytic part belongs to  $L^2(\mathbb{T})$ , then  $\Lambda_h(K) > 0$ .*

The result is perfectly sharp.

**Theorem 4.** *There exist a (non-trivial) pseudo-function  $S$  such that*

$$\sum_{n < 0} |\widehat{S}(n)|^2 < \infty \quad \Lambda_h(K) < \infty.$$

*Proof sketch.* Let  $K$  be a Cantor set on  $\mathbb{T}$  of the exact size, and let  $\mu$  be the natural probability measure on  $K$ . Denote by  $\mu$  also the harmonic extension of  $\mu$  into the disc, and let  $\tilde{\mu}$  be the conjugate harmonic function. Set

$$F(z) := e^{\mu + i\tilde{\mu}}.$$

In other words,  $1/F = e^{-\mu - i\tilde{\mu}}$  is an inner function (with the Blaschke term equal to 1). On the one hand, since  $\mu$  is singular,  $F$  is unbounded in the disk. On the other hand, its boundary value  $f$  is an  $L^2$  function — in fact the boundary values are exactly  $e^{i\tilde{\mu}}$  so are unimodular. The contrast between these two properties is what we will use.

First, one can prove that Taylor coefficients  $c(n)$  of  $F$  at 0 are of polynomial growth. So they correspond to an “analytic distribution” on  $\mathbb{T}$ :

$$G := \sum_{n \geq 0} c(n) e^{int}$$

On the other hand, the boundary value  $f$  is an  $L^2$  function

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{int}.$$

Consider the distribution  $S$  which is the difference  $G - f$ . It is supported by  $K$ . So it is left to get that  $S$  is a “pseudo-function”.

How can one get that  $\widehat{G}(n) \rightarrow 0$ ? One approach is to take the compact  $K$  very very large, and we took this path on our very first construction of an almost analytic singular pseudo-function [14]. But applying this with a compact  $K$  for which  $\Lambda_h(K) < \infty$  only gives that  $\widehat{G}(n)$  converge to 0 *in average*.

We solve this problem by letting the components of the Cantor set  $K$  on each step of its construction to “swim” a bit around their canonical positions, randomly. We perform this process heirarchically, moving each half of the set independently, then moving each quarter independently with respect to its position in the middle of its half, and so on. In other words, the position of an interval of order  $n$  is

the sum of  $n$  independent terms, one from each of its ancestors in the hierarchy defining the Cantor set.

It is well-known that a random perturbation “smoothes” the spectrum. For example, most of the constructions of Salem sets already mentioned are random, utilizing this effect, as once the spectrum has been smoothed the decay of the Fourier coefficients is the best one has given its  $l^2$  behaviour, which is known from Frostman’s theorem.

The proof has a number of features not shared by previous random constructions, though. Had we been interested in the Fourier coefficients of  $\mu$  itself, then we could have used the locality of the perturbation structure to write  $\hat{\mu}(n)$  as a sum of *independent* terms and apply, say, Bernstein’s inequality. But we are interested in  $e^{\mu+i\tilde{\mu}}$  and  $\tilde{\mu}$ , the complex conjugate of  $\mu$  does not have a local structure. Moving a small piece of  $K$  even a little will change  $\tilde{\mu}$  throughout.

Another difficulty is that, no matter what probabilistic model one were to take, it is always true that

$$\sum \mathbb{E}|\hat{G}(n)|^2 = \infty.$$

This is because, has this sum been finite we would have that  $G$  is in  $l^2$  almost surely, which is impossible. Hence we are forced to work with 4<sup>th</sup> moments. Thus we show that  $\sum \mathbb{E}|\hat{G}(n)|^4 < \infty$ , and conclude that  $\hat{G}(n) \rightarrow 0$ , almost surely, proving the theorem. The interested reader can see the details of the calculation of  $\mathbb{E}|\hat{G}(n)|^4$  in [16], or in [15] where a similar random construction was used.  $\square$

Notice that theorems 3 and 4 give a sharp estimate for the size of the exceptional set of a non-classic analytic expansion.

A couple of words about the smoothness. It was proved in [15] that the non-analytic half of  $S$  in Theorem 2 can be infinitely smooth, that is  $\hat{S}(-n) = o(1/n^k)$  for every  $k$ . In fact, we have a precise description of the exact maximal smoothness achievable, or in other words, a quasi-analyticity-like result. Surprisingly, while in classical quasi-analyticity results the “critical smoothness” is approximately  $\hat{S}(n) \asymp e^{-n/\log n}$ , i.e. quite close to analytical smoothness, in our case the maximal smoothness allowable is approximately  $\hat{S}(-n) \asymp e^{-\log n \log \log n}$  i.e. only slightly above  $C^\infty$  smoothness.

It seems that one has to pay for smoothness by the size of support, which probably must increase.

*Question.* How does the “critical size” of the support of  $K$  depend on the order of smoothness?

## 5. NON-SYMMETRY FOR MEASURES

Theorem 4 reveals a *non-symmetry* phenomenon for *distributions*. A singular complex measure can not be “almost analytic” according to the theorem of the brothers Riesz. Measures in general have more symmetry. Example (Rajchman [24] or [11, §1.4]): If the Fourier coefficients of a complex measure on the circle are  $o(1)$  at positive infinity then the same is true at negative infinity. Another result of this sort (Hruščev and Peller [6], Koosis and Pichorides, see [13]):

**Theorem.** *If  $\sum_{n>0} |\hat{\mu}(n)|^2 / |n| < \infty$ , then the same is true for sum over  $n < 0$ .*

However, certain non-symmetry may happen even for singular measures.

**Theorem 5.** Given  $d > 0$ ,  $p > 2/d$  there is a measure  $\mu$  supported on a compact  $K$  such that

- (i)  $\dim K = d$ ;
- (ii)  $\widehat{\mu} \in l_p(\mathbb{Z}^-)$  but not in  $l^p(\mathbb{Z}^+)$

See [16, theorem 3.1].

*Remarks.* 1. The restriction on  $p$  is sharp, due to theorem 2.

2. Not every  $K$  may support such a measure,  $K$  has to be a Salem set (in the same sense as above).

*Question.* Let  $K$  supports  $S$  with  $\widehat{S}(n) \rightarrow 0$  as  $n \rightarrow \infty$ . Does it support an  $S'$  with the two-sides condition  $|n| \rightarrow \infty$ ? (in other words, can a  $\mathcal{U}$ -set support a distribution with  $\widehat{S}(n) \rightarrow 0$  as  $n$  tends to positive  $\infty$ ?)

## 6. "ARITHMETICS" OF COMPACTS

It was already mentioned that not only the size of the support but rather its arithmetic nature may play a crucial role in the behaviour of measures or distributions. The most remarkable illustration of this phenomenon is related to the following problem. Let  $K_\theta$  be the symmetric Cantor set with dissection ratio  $\theta$ , i.e. the set one gets by starting with a single interval and then repeatedly replacing each interval of length  $a$  by two intervals of length  $\theta a$ . When may  $K_\theta$  satisfy the "Menshov property", that is may support a measure or distribution with Fourier transform vanishing at infinity? It was Nina Bari who discovered (in the case when  $\theta$  is rational) that the answer has nothing to do with the size of the compact and depends on arithmetics of  $\theta$ . This result is not so often mentioned in western literature. The much deeper Salem-Zygmund theorem extends this phenomenon to irrational  $\theta$ . The result is that  $K_\theta$  supports a null-measure if and only if the dissection ratio is not the inverse of a Pisot number, i.e. an algebraic integer  $> 1$  all whose algebraic conjugates are  $< 1$  in absolute value.

Relations between uniqueness theory and number theory are by now a well-developed theory, see the book [21]. Let us also recall the deterministic constructions of Salem sets [10], the role of the so-called Kronecker sets [7, chapter 7] and results on the set of non-normal numbers [22, 8, 19].

We'll finish this survey by a recent result in which the arithmetics of a compact also plays a crucial role. It relates to the so-called Wiener problem on cyclic vectors.

**Definition.** Let  $1 \leq p \leq \infty$ . A function  $x \in l^p(\mathbb{Z})$  is called a cyclic vector if its translates spans the whole space.

Wiener characterized cyclic vectors for  $p = 1$  and  $p = 2$ :

**Theorem.**

- (i)  $x = \{x_k\}$  is cyclic in  $l^1$  iff  $\widehat{x}(t) := \sum x_k e^{ikt}$  has no zeros;
- (ii)  $x$  is cyclic in  $l^2$  iff  $\widehat{x}(t) \neq 0$  a.e.

Wiener conjectured [25, page 93] that for every  $p$ , or at least for  $p < 2$ , cyclic vectors can be characterized by a certain "negligibility" condition of the zero set of  $\widehat{x}$ . It turned out this is not the case:

**Theorem 6 ([18]).** Let  $1 < p < 2$ . Then there are two vectors  $x$  and  $y \in l^1$  such that

- (i) The zero sets of  $\widehat{x}$  and  $\widehat{y}$  are the same;

(ii)  $x$  is cyclic in  $l^p$ , while  $y$  is not.

The approach to the proof is based on certain development of ideas of Piatetskii-Shapiro [22] who apparently was the first who inserted functional-analytic and probabilistic ideas into Riemannian uniqueness theory.

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GK: DEPARTMENT OF MATH, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL.

AO: SCHOOL OF MATHEMATICS, TEL AVIV UNIVERSITY, TEL AVIV 69978, ISRAEL.

*E-mail address:* gady.kozma@weizmann.ac.il, olevskii@post.tau.ac.il